MATHEMATICS METHODS

MAWA Semester 1 (Unit3) Examination 2016

Calculator-Assumed

Marking Key

Section Two: Calculator-assumed (98 Marks)

Question 10

Solution

$$\frac{dV}{dr} = 2\pi r^2$$

$$\frac{\delta r}{r} \approx \frac{1}{2\pi r^2} \times \frac{\delta V}{r}$$

$$= \frac{1}{3} \times \frac{3}{2\pi r^3} \times \delta V$$

$$= \frac{1}{3} \times \frac{\delta V}{V}$$

$$= \frac{1}{3} \times \frac{1.5}{100}$$

$$= 0.005 \times 100 = 0.5\%$$

OR

$$V = \frac{2}{3}\pi r^3$$

$$\delta V \approx 2\pi r^2 \delta r$$

$$\frac{\delta V}{V} = \frac{2\pi r^2}{\frac{2}{3}\pi r^3}\delta r = 3\,\frac{\delta r}{r}$$

$$\therefore \frac{\delta r}{r} = \frac{0.015}{3} = 0.005 \times 100 = 0.5\%$$

Marking key/mathematical behaviours	Marks
• states the correct volume	1
• uses incremental formula correctly	1
• writes the incremental formula as ratios	1
• calculates the correct percentage change	1

Question 11(a)

Solution

y	1	2	3
P($Y = y$)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Marking key/mathematical behaviours	Marks
• calculates both probabilities correctly	1

Question 11(b)(i)

Solution

$$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Marking key/mathematical behaviours	Marks
• calculates correct probability	1

Question 11(b)(ii)

Solution $\frac{1}{2}$	
Marking key/mathematical behaviours	Marks
• calculates correct probability	1

Question 11(b)(iii)

Solution $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$	
Marking key/mathematical behaviours	Marks
• calculates correct probability	1

Question 11(b)(iv)

Solution $\frac{1}{3} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{6} = \frac{7}{18}$	
Marking key/mathematical behaviours	Marks
• indicates both faces being 1, 2 or 3 • calculates correct probability	1 1

Question 11(b)(v)

Solution Possible pairings are 1 3 or 3 1 or 2 2 $\frac{1}{3} \times \frac{1}{6} \times 2 + \frac{1}{2} \times \frac{1}{2} = \frac{13}{36}$	
Marking key/mathematical behaviours	Marks
• gives correct pairings including both possibilities for 1 and 3 • calculates correct probability	1, 1 1

Question 12(a)

Solution

$$\frac{dy}{dx} = -4axe^{x^2}$$

$$0 = -4axe^{x^2}$$

$$x = 0$$

when $x = 0$,

$$y = a - 2ae^0$$

$$y = -a$$

stationary point at $(0, -a)$

Marking key/mathematical behaviours	Marks
• determines the derivative using the chain rule	1
• equates to zero and solves	1
• substitutes to determine y-coordinate	1

Question 12(b)

Solution

$$\frac{d^2y}{dx^2} = -4ax(2xe^{x^2}) - 4ae^{x^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = -4a$$

Since a is a positive constant the second derivative is negative.
It is a maximum

Marking key/mathematical behaviours	Marks
• determines the first and second parts of the second derivative using the product rule and chain rule	1,1
• determines the value of the second derivative when x=0	1
• states the nature of the stationary point	1

Question 13(a)

Solution

$$\frac{dy}{dx} = x\cos(x) + \sin(x)$$

Marking key/mathematical behaviours	Marks
• correctly differentiates using the product rule	1,1

Question 13(b)

Solution

$$\int \frac{dy}{dx} = \int x\cos(x) + \sin(x)\,dx$$
$$x\sin(x) = \int x\cos(x)\,dx + \int \sin(x)\,dx$$
$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx$$
$$= x\sin(x) + \cos(x) + c$$

Marking key/mathematical behaviours	Marks
• integrates both sides of the derivative obtained in part (a)	1
• replaces the LHS by y	1
• rearranges correctly	1
• integrates $\sin(x)$ correctly	1

Question 14(a)

Solution

x	4	6	8	9	11	14
$\mathrm{P}(X = x)$	0.16	**0.32**	**0.16**	**0.16**	**0.16**	0.04

Marking key/mathematical behaviours	Marks
• calculates correct probability for each score	1, 1, 1, 1

Question 14(b)

Solution

y	–6	**–4**	**–2**	**–1**	**1**	4
$\mathrm{P}(Y = y)$	0.16	**0.32**	**0.16**	**0.16**	**0.16**	0.04

Marking key/mathematical behaviours	Marks
• correctly completes distribution table	1, 1, 1

Question 14(c)

Solution

The sum of $y \times \mathrm{P}(Y = y) = -2.40$ cents
For 50 games = –240 cents which is a loss of $2.40

Marking key/mathematical behaviours	Marks
• correctly calculates expected value	1
• multiplies by 50	1
• states that it is a loss	1

Question 15

Solution

$$\frac{d}{dx}\int_a^x (\ f(t)+e^t\ )\ dt - 2\int_0^x \frac{d}{dt}\ (f(t)+e^{2t}\ )\ dt = 2$$

$$f(x)+e^x-2(f(x)+e^{2x}-f(0)-1\ )=2$$

$$-f(x)+e^x-2e^{2x}+4=2$$

$$f(x)=2+e^x-2e^{2x}$$

Marking key/mathematical behaviours	Marks
• applies the fundamental theorem to first integral	1
• evaluates second integral	1
• expands brackets correctly	1
• substitutes *f(0)*	1
• rearranges for *f(x)* correctly	1

Question 16

Solution

$$\int_0^k (\sqrt{k}-\sqrt{x})^2\,dx$$

$$=\int_0^k k-2\sqrt{k}\,x^{\frac{1}{2}}+x\ dx$$

$$=\left[\ kx-\frac{4}{3}\sqrt{k}\,x^{\frac{3}{2}}+\frac{x^2}{2}\right]_0^k$$

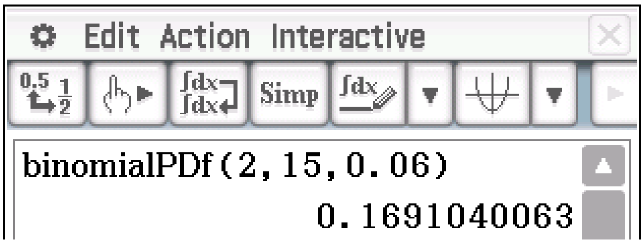
$$=k^2-\frac{4}{3}k^{\frac{1}{2}}k^{\frac{3}{2}}+\frac{k^2}{2}-0$$

$$=\frac{k^2}{6}$$

Marking key/mathematical behaviours	Marks
• correctly expands brackets	1
• correctly integrates	1
• correctly substitutes limits	1
• correctly simplifies	1

Question 17(a)

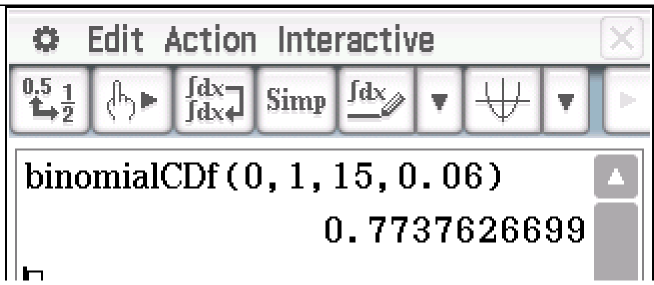
Solution



Marking key/mathematical behaviours	Marks
• determines correct probability	1

Question 17(b)

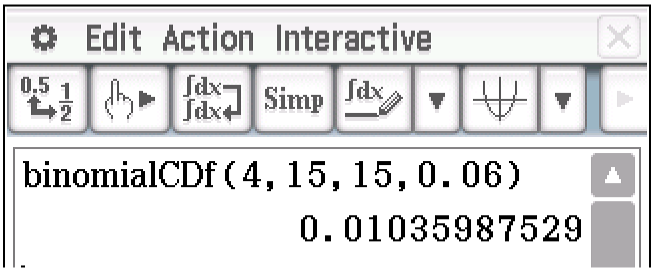
Solution



Marking key/mathematical behaviours	Marks
• determines correct probability	1

Question 17(c)

Solution

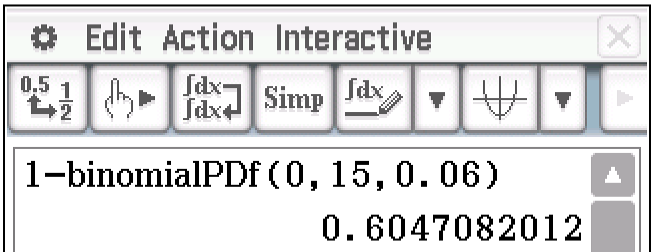


Marking key/mathematical behaviours	Marks
• determines correct probability	1

Question 17(d)

Solution

P(at least 1)=1 – P(0)



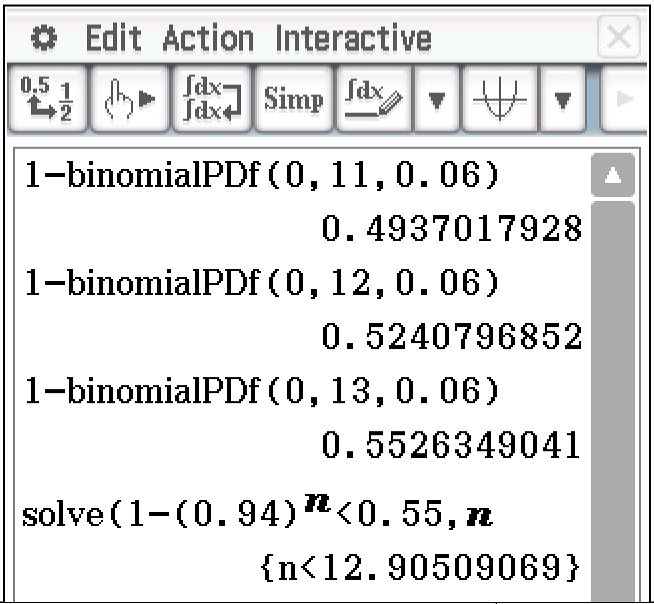
Marking key/mathematical behaviours	Marks
• correctly uses complementary event	1
• determines correct probability	1

Question 17(e)

Solution

Using P(at least 1)=1 – P(0) and testing n = 11, 12, 13. Largest sample is 12.

OR using solve



Marking key/mathematical behaviours	Marks
• correctly uses complementary event and tests 11, 12, 13	1,1
• determines correct sample size	1

Question 18(a)

Solution

In triangle, height $= 2\cos\theta$ and base $= 2\sin\theta$

V = area of trapezium $\times 8$

$$= \frac{2\cos\theta}{2}\times\left(2+2+2\times 2\sin\theta\right)\times 8$$

$$= \cos\theta\times\left(4+4\sin\theta\right)\times 8$$

$$= 32\cos\theta\left(1+\sin\theta\right)$$

Marking key/mathematical behaviours	Marks
• identifies height and base of triangle	1
• uses suitable formula for area of base	1
• simplifies and factorises result	1

Question 18(b)

Solution

$$\text{solve}\left(10=32\cdot\cos(\theta)\cdot(1+\sin(\theta)),\theta,0,0,\frac{\pi}{2}\right)$$

$$\{\theta=1.412913449\}$$

Marking key/mathematical behaviours	Marks
• recognises to equate the volume equation to 10 • solves for theta	1 1

Question 18(c)

Solution

$V'(\theta)=-32\sin\theta(1+\sin\theta)+32\cos\theta\cos\theta$

For max: $V'(\theta)=0\Rightarrow\theta=0.52$

$V''(0.52)=-83.14\Rightarrow$ maximum

$\text{V}(0.52)=41.57\text{ m}^3$

Maximum capacity is 41570 kL

Marking key/mathematical behaviours	Marks
• determines the first part of the derivative using the product rule • determines the second part of the derivative using the product rule • equates derivative to zero and solves for theta • justifies maximum • determines the volume • states the capacity	1 1 1 1 1 1

Question 19(a)

Solution

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	0.5	**0.54**	**0.67**	1

Rectangle	$0 - \frac{\pi}{6}$	$\frac{\pi}{6} - \frac{\pi}{3}$	$\frac{\pi}{3} - \frac{\pi}{2}$	Total
Lower rectangle area	0.26	**0.28**	**0.35**	**0.89**
Upper rectangle area	0.28	**0.35**	**0.52**	**1.15**
			Mean	**1.02**

Marking key/mathematical behaviours	Marks
• correctly calculates the values of y	1,1
• correctly calculates areas of upper and lower rectangles	1
• correctly calculates totals	1
• correctly calculates mean	1

Question 19(b)

Solution

$$\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) = \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$
$$= \frac{(\cos x + 1)}{(1+\cos x)^2}$$
$$= \frac{1}{1+\cos x}$$

Marking key/mathematical behaviours	Marks
• correctly differentiates using the quotient rule	1
• expands brackets and simplifies	1
• uses the given information correctly and obtains solution	1

Question 19(c)

Solution

$$\text{Area} = \int_0^{\frac{\pi}{3}} f(x)\,dx = \left[\frac{\sin x}{1+\cos x}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}}$$

$$= \frac{\sqrt{3}}{3}$$

Marking key/mathematical behaviours	Marks
• correctly uses part (c) for the integral	1
• evaluates the integral	1
• simplifies solution	1

Question 20(a)

Solution

$\text{solve}\left(0=2\cdot e^{2\cdot t}-10, t\right)$

$\{t=0.8047189562\}$

Marking key/mathematical behaviours	Marks
• recognises that the particle is at rest when $v = 0$	1
• Solves for t.	1

Question 20(b)

Solution

$$a(t) = \frac{dv}{dt}$$
$$= 4e^{2t}$$
$$a(0) = 4(1)$$
$$= 4 \text{ m/s}^2$$

Marking key/mathematical behaviours	Marks
• determines the derivative of the velocity function	1
• determines the acceleration when $t = 0$.	1

Question 20 (c)

Solution

$$x(t) = \int (2e^{2t} - 10)dt$$
$$= e^{2t} - 10t + c$$
$$3 = e^0 + c$$
$$c = 2$$
$$x(2) = e^4 - 20 + 2$$
$$= 36.60m$$

Marking key/mathematical behaviours	Marks
• integrates *v(t)* to obtain general rule for *x(t)*	1
• uses the initial conditions to determine *c*	1
• calculates the displacement at *t=2*	1

Question 20 (d)

Solution

$$x(t) = \int_0^4 \left|2e^{2t} - 10\right| dt$$
$$= 2948.05 \text{ m}$$

OR

$$x(0) = 3$$
$$x(0.80) = -1.05$$
$$x(4) = 2942.96$$
$$DIST = 4.05 + 1.05 + 2942.96$$
$$= 2948.05 \text{ m}$$

Marking key/mathematical behaviours	Marks
• recognises the need for absolute value	1
• identifies the limits as 0 and 4	1
• determines the distance travelled	1

Question 21 (a)

Solution	
Define $P(t)=1500\cdot e^{0.07\cdot t}$ done P(3) 1850.51709	
Marking key/mathematical behaviours	Marks
• writes the function for the population • determines the population when $t=3$	1 1

Question 21 (b)

Solution	
solve(P(t)=2000, t) {t=4.109743892} During 2014	
Marking key/mathematical behaviours	Mark
• determines the value of t when $P = 2000$ • states the year this occurs	1 1

Question 21(c)

Solution	
P(6) 2282.942333 Define $Q(t)=P(6)\cdot e^{-0.05\cdot t}$ done solve(1500=Q(t), t) {t=8.4} During May 2024	
Marking key/mathematical behaviours	Marks
• determines the population at the start of 2016 • states an equation for the new population • equates this new equation to 1500 and solves for t • states the month and year corresponding to this value of t	1 1 1 1

Question 22

Solution
Height of triangle OPR = $ah^2 + bh$ Area under curve from 0 to h = $\int_0^h ax^2 + bx\,dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^h = \frac{ah^3}{3} + \frac{bh^2}{2}$ Equation of line OQ : $\frac{d}{dx}(ax^2 + bx)\Big\|_{x=0} = b$ $\therefore$ equation of line OQ is $y = bx$ Area of triangle OPR : $\frac{h}{2}(ah^2 + bh) = \frac{ah^3}{2} + \frac{bh^2}{2}$ Area of triangle OQR : $\frac{h}{2} \times bh = \frac{bh^2}{2}$ Area of region A : $\frac{ah^3}{2} + \frac{bh^2}{2} - \left(\frac{ah^3}{3} + \frac{bh^2}{2}\right) = \frac{ah^3}{6}$ Area of region B : $\frac{ah^3}{3} + \frac{bh^2}{2} - \frac{bh^2}{2} = \frac{ah^3}{3}$ Ratio of region A to region B : $\dfrac{\frac{ah^3}{3}}{\frac{ah^3}{6}} = 1:2$

Marking key/mathematical behaviours	Marks
• determines height of triangle OPR in terms of h	1
• determines area under curve	1
• determines equation of line OQ	1
• determines areas of triangles OPR and OQR	1
• determines area of region A and B	1
• calculates ratio of region A to region B	1

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